HARMONIC OSCILLATOR INTERACTION WITH SQUEEZED RADIATION

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Although the problem of the electromagnetic radiation by a quantum harmonic oscillator is considered in textbooks on quantum mechanics (see, e.g., [1]) some its aspects seem to be not clarified until now. By this we mean that usually the initial quantum states of both the oscillator and the field are assumed to be characterized by a definite energy level of the oscillator and definite occupation numbers of the field modes. In connection with growing interest in squeezed states it would be interesting to analize the general case when the initial states of both subsystems are arbitrary superpositions of energy eigenstates. This problem was considered partly in Refs. 2-4, where the power of the spontaneous emission was calculated in the case of an arbitrary oscillator's initial state (but the field was supposed to be initially in a vacuum state). In the present article we calculate the rate of the oscillator average energy and squeezing and correlation parameter change under the influence of an arbitrary external radiation field. Some other problems relating to the interaction between quantum particles (atoms) or oscillators with the electromagnetic radiation being in arbitrary (in particular, squeezed) state were investigated, e.g., in Refs 5-7.

Let us describe a charged harmonic oscillator by a Hamiltonian

$$H_o = \hbar\omega\, a^+ a \qquad (1)$$

and the field by a hamiltonian

$$H_R = \hbar \sum_j \omega_j b_j^+ b_j . \qquad (2)$$

here ω is the frequency of the oscillator, ω_j - ones of field modes, a,b - corresponding destruction operators.

In a rather general case interaction can be described in a form

$$H_i = \hbar \sum_j \left(\mu_j a^+ b_j^+ + \lambda_j a^+ b_j + H.c. \right) \qquad (3)$$

(H.c. means hermitian conjugated part, μ_j and λ_j are constants).

In Schrodinger picture an arbitraty initial state vector $|\psi(0)\rangle$ evolves into a state vector $|\psi_S(t)\rangle$ as predicted by Schrodinger equation with Hamiltonian $H = H_0 + H_R + H_i$.

In interaction picture any Schrodinger operator Q changes according to evolution operator U_0 corresponding to $H = H_0 + H_R$

$$Q(t) = U_0^+(t) Q U_0(t) . \qquad (4)$$

For example

$$a(t) = a \exp(-i\omega t), \qquad b_j(t) = b_j \exp(-i\omega_j t) . \qquad (5)$$

The interaction Hamiltonian in this picture

$$H_I = \hbar \sum_j \left(\mu_j a^+ b_j^+ \exp(i\omega_j t + i\omega t) + \lambda_j a^+ b_j \exp(-i\omega_j t + i\omega t) + H.c. \right) \qquad (6)$$

generates evolution operator U(t) so that a state vector in this picture defined as

$$|\psi(t)\rangle = U_0^+(t) |\psi_S(t)\rangle \qquad (7)$$

will variate according to

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle . \qquad (8)$$

Expectation value in this picture

$$\langle Q \rangle_I = \langle \psi(t) | Q | \psi(t) \rangle \qquad (9)$$

variates slowly, only due to interaction. On the other hand, it is related to the conventional expectation value as follows

$$\langle Q \rangle_I = \langle \psi_S | U_0 Q U_0^+ | \psi_S \rangle . \qquad (10)$$

After introducing designations we can pose several questions to answer:

1. Can absorption and emission be distinguished in a general case ?
2. Then how to calculate the rates of these processes ?

3. Is time ordering important in perturbation calculation for this case ?

4. Does stimulated emission manifest itself ?

5. How does squeezing parameters of the oscillator and the field vary ?

To calculate the rates of the processes we need to consider infinitely long time intervals $\tau \to \infty$ in comparison with oscillation period. But they must be much shorter than damping time. Then the evolution operator has meaning of scattering matrix S transforming initial state $|\psi(0)\rangle \equiv |i\rangle$ to resulting one $|r\rangle$. From Heisenberg equation one gets

$$S = \exp_T(-iT), \tag{11}$$

where all products are believed time-ordered (designated with subscript T), and T - matrix is given by

$$T = \int_{-\infty}^{\infty} H_I(t)/\hbar \, dt. \tag{12}$$

For our particular case

$$T = 2\pi\hbar \sum_j \left(\lambda_j a^+ b_j \delta(\omega_j-\omega) + H.c. \right), \tag{13}$$

here the terms with μ vanish because of a factor $\delta(\omega_j+\omega)$. Further $\delta_j \equiv \delta(\omega_j-\omega)$. Delta function originates as a limit of an integral

$$Int = \int_{-\tau/2}^{\tau/2} \exp(i\Omega t) \, dt \tag{14}$$

(here the initial instant in time re-designated as $-\tau/2$). Limits of this integral are

$$Int \to \tau, \qquad \text{if } \Omega \to 0, \tag{14'}$$

$$Int \to 2\pi \, \delta(\Omega), \text{ if } \tau \to \infty. \tag{14"}$$

Conventional techique in quantum electrodynamics is as follows [8]. T - matrix is splitted into two parts - absorption part

$$T^- = 2\pi \sum_j \lambda_j a^+ b_j \delta(\omega_j-\omega) \tag{15}$$

and hermitian conjugated emission part T^+. Then probability for

time τ of absorption (and similarly emission) is declared as

$$P = \langle i|T^+T^-|i\rangle \equiv \sum_f |\langle f|T^-|i\rangle|^2, \qquad (16)$$

where summation is performed over a complete set of possible final states. If rewritten in a form

$$T^- = 2\pi M^- \delta(E_f - E_i), \qquad (17)$$

where E_f and E_i are energies of final and initial states, it shows employing (14) that (16) expresses the well-known Fermi's rule

$$P = \frac{2\pi\tau}{\hbar} \sum_f |\langle f|M^-|i\rangle|^2 \, \delta(E_f - E_i). \qquad (18)$$

But is it always valid and why probability is defined in this manner ?

The expansion of S - matrix (11) is as follows

$$S = 1 - i(T^+ + T^-) - (T^2)_T/2 + \ldots \qquad (19)$$

The identity of normalization must be valid in all orders of perturbation, i.e. for all powers of T as it is proportional to the first power of coupling constant :

$$1 = \langle r|r\rangle = \langle i|i\rangle + \langle i|T^+T^+|i\rangle + \langle i|T^-T^+|i\rangle + \langle i|T^+T^-|i\rangle + \langle i|T^-T^-|i\rangle - \langle i|(T^2)_T|i\rangle + \ldots \qquad (20)$$

Then terms from second to fifth can be interpreted as a probability of transitions in the second order, since the first and the sixth will be probability to stay in the initial state. So conventional procedure ignores the second and the fifth terms. It is possible only if $T^-|i\rangle$ is orthogonal to $T^+|i\rangle$. It can happen when either field or the oscillator is in energy eigenstate. Then actually only two levels are involved in any sort of transitions. In this case emission and absorption can be distinguished. That is on obtaining after measurement one of $|f\rangle$ states we can tell a result of absorption from a result of emission.

For arbitrary initial state they cannot be distinguished experimentally. But the total probability of emission and absorption together in (20) does not have physical meaning. Therefore we have to revise our approach. More well-grounded

procedure is to calculate not probabilities but observable variations :

$$\Delta\langle Q\rangle_I = \langle i|S^+QS|i\rangle - \langle i|Q|i\rangle. \qquad (21)$$

Besides, we do not need to introduce the Fock basis $|f\rangle$, but deal only with the initial state.

Since the observable variation is expected to grow with time, to calculate the rates of the processes we need to consider only terms proportional to long time τ. We will see later that expressions like (21) contain terms with factor $\delta(\omega_j-\omega)$ and terms with $\delta^2(\omega_j-\omega)$ under a sign of summation. One power of delta function disappear because of summation over the continuum of modes. The rest one power will transform to factor τ. So terms with delta function of infinitely little difference to the first and zeroth powers will give non-growing with time observable variation. Consequently, these terms represent dressing bare states by virtual quanta. Terms with the second powers of delta function will give time-proportional variations of observables. Just these terms correspond to transitions with creation of real quanta.

For our case we need S - matrix up to the second order of perturbation. In this order a time-ordered product

$$(T^2)_T = \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \left[H_I(t_1)H_I(t_2)\right]_T/\hbar^2, \qquad (22)$$

where

$$\left[H_I(t_1)H_I(t_2)\right]_T = \begin{cases} H_I(t_2)H_I(t_1), & \text{if } t_2 > t_1, \\ H_I(t_1)H_I(t_2), & \text{if } t_1 > t_2, \end{cases} \qquad (23)$$

is different from non-ordered product

$$(T^2)_T = T\times T + T^2_{dif} \qquad (24)$$

by a term

$$T^2_{dif} = \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \left[H_I(t_2)H_I(t_1)\right]/\hbar^2, \qquad (25)$$

The latter expression depends on time like

$$\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \exp(i\Lambda t_2 + i\Omega t_1) = 4\pi\,\delta(\Lambda+\Omega)\lim_{z\to\infty}\frac{\exp[i(\Lambda-\Omega)z]-1}{i(\Lambda-\Omega)}. \qquad (26)$$

Such terms do not vanish only if $\Lambda = -\Omega$. If $\Lambda = \omega + \omega_j$ then the last factor in (26) is not singular. Terms with $\Lambda = \omega_j - \omega$ or opposite give a contribution to T^2_{dif}

$$\sum |\lambda_j|^2 [ab_j^+, a^+b_j] \frac{\exp(2i\Lambda Z)-1+\exp(-2i\Lambda Z)-1}{2i\Lambda}, \qquad (27)$$

which is not singular either. So T^2_{dif} contains first powers of delta functions and can be neglected compared to T (the former is coupling constant λ_j times less).

We arrive to an assumption

$$S = 1 - iT - T\times T/2, \qquad (28)$$

that leads to ,

$$\Delta\langle Q\rangle_I = \langle i \mid i[T,Q] - \frac{1}{2}[T,[T,Q]] \mid i\rangle \qquad (29)$$

the first term being virtual and the second - real. Straightforward calculation using (29) gives for example

$$\Delta\langle a\rangle_I = -\sum_j i\lambda_j 2\pi\delta_j\langle b_j\rangle - \frac{1}{2}\sum_j |\lambda_j|^2(2\pi\delta_j)^2\langle a\rangle, \qquad (30a)$$

$$\Delta\langle b_k\rangle_I = -i\lambda_k^* 2\pi\delta_k\langle a\rangle - \frac{1}{2}\lambda_k^* 2\pi\delta_k \sum_j \lambda_j 2\pi\delta_j\langle b_j\rangle, \qquad (30b)$$

$$\Delta\langle a^+a\rangle_I = i\sum_j (\lambda_j^*\langle b_j^+a\rangle - \lambda_j\langle a^+b_j\rangle)2\pi\delta_j - \sum_k |\lambda_k|^2(2\pi\delta_k)^2\langle a^+a\rangle$$
$$+ \frac{1}{2}\sum_{k,j} (\lambda_j\lambda_k^*\langle b_k^+b_j\rangle + \lambda_j^*\lambda_k\langle b_j^+b_k\rangle)\ (2\pi)^2\delta_j\delta_k\ . \qquad (30c)$$

$$\Delta\langle b_k^+b_k\rangle_I = i(\lambda_k\langle a^+b_k\rangle - \lambda_k^*\langle b_k^+a\rangle)2\pi\delta_k + |\lambda_k|^2\langle a^+a\rangle(2\pi\delta_k)^2$$
$$- \pi\delta_k\left(\lambda_k\sum_j \lambda_j^* 2\pi\delta_j\langle b_j^+b_k\rangle + \lambda_k^*\sum_j \lambda_j 2\pi\delta_j\langle b_k^+b_j\rangle\right). \qquad (30d)$$

These variations are expressed in terms of expectation values in the initial state (designated with triangle brackets). One can define quadrature component variances by

$$D(P,Q) = \frac{1}{2}\left(\langle PQ\rangle + \langle QP\rangle\right) - \langle P\rangle\langle Q\rangle. \qquad (31)$$

Their variations can be expressed similar to

$$\Delta D(a,a)_I = \Delta\langle aa\rangle_I - 2\langle a\rangle\ \Delta\langle a\rangle_I - (\Delta\langle a\rangle_I)^2. \qquad (32)$$

This kind of variance is important because in canonical

coordinate-momentum space ImD(a,a) corresponds to correlation and ReD(a,a) - to squeezing. In Schrodinger picture they rapidly transfer from each to other.

Retaining in (30) and (32) only terms proportional to τ and dividing by τ we obtain time derivative equations. From them we clearly see that radiation damping

$$\gamma = \sum_j |\lambda_j|^2 2\pi\delta_j \tag{33}$$

determines variation of amplitudes

$$\frac{d}{dt} \langle a \rangle_I = - \frac{\gamma}{2} \langle a \rangle, \tag{34a}$$

$$\frac{d}{dt} \langle b_k \rangle_I = - \frac{1}{2} |\lambda_k|^2 2\pi\delta_k \langle b_k \rangle. \tag{34b}$$

These equations coincide with those obtained usually in the frame of Wigner - Weisskopf approximation. Field modes and the oscillator exchange their energies. As a result there is no effect of stimulated emission but only two independent fluxes of energy:

$$\frac{d}{dt} \langle a^+ a \rangle_I = - \gamma \langle a^+ a \rangle + \sum_k |\lambda_k|^2 2\pi\delta_k \langle b_k^+ b_k \rangle, \tag{34c}$$

$$\frac{d}{dt} \langle b_k^+ b_k \rangle_I = |\lambda_k|^2 2\pi\delta_k \langle a^+ a \rangle + |\lambda_k|^2 2\pi\delta_k \langle b_k^+ b_k \rangle. \tag{34d}$$

Squeezing-correlation parameter behaves in a similar way :

$$\frac{d}{dt} D(a,a)_I = - \gamma D(a,a) - \sum_k \lambda_k^2 2\pi\delta_k D(b_k,b_k), \tag{34e}$$

$$\frac{d}{dt} D(b_k,b_k)_I = - \lambda_k^{*2} 2\pi\delta_k D(a,a) - |\lambda_k|^2 2\pi\delta_k D(b_k,b_k). \tag{34f}$$

Further development can be made for the specific expressions of coefficients in Hamiltonian (3). For the continuum of modes summation is substituted by integration over phase space and summation over polarization indexes r

$$\sum_j \rightarrow \sum_r \int V\rho \, d\omega \, d\Omega \tag{35}$$

with volume V, solid angle element $d\Omega$, mode frequency density

$$\rho = \frac{\omega^2}{8\pi^3 c^3} . \tag{36}$$

Decomposition of vector potential A(r,t) over mode variables is

$$A(r,t) = \sum_j \sqrt{\frac{\hbar}{2\omega_j \varepsilon_o V}}\, e_j \,(b_j(t)\exp(ik_j r) + H.c.). \qquad (37)$$

where e_j is a polarization vector and k_j - a wave vector of j-th mode.

Gauge invariance substitution of oscillator momentum $p \to p - eA$ leads to the interaction Hamiltonian (3)

$$H_i = \frac{-epA}{m} + \frac{e^2A^2}{2m}. \qquad (38)$$

Here e,m are the charge and the mass of the oscillator. The second term in this case proves to be a unity operator in state-space of the oscillator. Hence it results in an infinitely little renormalization of field energy because of a factor 1/V (for infinitely large volume V). The coupling constant will be

$$\lambda_j = -i\,\frac{e}{2}\sqrt{\frac{\omega}{\omega_j m \varepsilon_o V}}\cos\theta_j, \qquad (39)$$

where θ_j is the angle between a polarization vector and the oscillation direction.

On the other hand, from the Hamiltonian in another gauge form

$$H_i = -eqE, \qquad (40)$$

where q is a coordinate of the oscillator and E is the electric field vector, it follows that the coupling constant

$$\lambda'_j = \lambda_j \frac{\omega_j}{\omega}. \qquad (41)$$

But as all expressions contain delta functions $\delta(\omega_j-\omega)$, constants (39) and (41) coincide. We see that it is one of the cases when gauge transform, performed over state vectors in the absence of vector potential and corresponding to a change from gauge form (40) to (38), does not make any difference. These transforms were considered in detail in Ref. 10.

Einstein's stimulated coefficient can be also introduced. However it is different from a common one - it depends on the angle and expresses radiation power instead of probability :

$$B = 2\pi V|\lambda|^2 = \frac{\pi e^2 \cos^2\theta}{2m\varepsilon_o} . \qquad (42)$$

The spontaneous emission coefficient is obtained from (33). Integration should be performed over solid angles of polarisation vectors (they are also isotropically distributed), not wave vectors of modes :

$$\gamma = \frac{\omega^2}{4\pi^3 c^3} \int B \, d\Omega = \frac{e^2\omega^2}{6\pi m\varepsilon_o c^3} . \qquad (43)$$

A light beam containing several close modes has an energy density

$$W_\omega = \sum_r \rho \langle b^+ b \rangle \hbar\omega \qquad (44)$$

$$\text{or} \qquad W = \int W_\omega \, d\Omega. \qquad (45)$$

It will allow us to express eqs.(34) through physically meaningfull values.

$$\frac{d}{dt}\left[\hbar\omega\langle a^+a\rangle\right] = -\gamma\hbar\omega\langle a^+a\rangle + \int BW_\omega d\Omega, \qquad (46a)$$

$$\frac{d}{dt}\left[WV\right] = B\left[\rho\hbar\omega\langle a^+a\rangle - W_\omega\right], \qquad (46b)$$

$$\frac{d}{dt} D(a,a) = -\gamma D(a,a) + \int BW_\omega \frac{D(b,b)}{\hbar\omega\langle b^+b\rangle} d\Omega, \qquad (46c)$$

$$\frac{d}{dt}\left[WVD(b,b)\right] = B\left[\rho\hbar\omega\langle b^+b\rangle D(a,a) - W_\omega D(b,b)\right]. \qquad (46d)$$

All above discussed enables us to answer posed questions :

1. In general absorption and emission can not be distinguished.
2. So not Fermi's rule but expectation values should be used to calculate the rates of these processes.
3. Time ordering in this case is not important up to the second order of perturbation.
4. Stimulated emission does not manifest itself in the final result.
5. Energy and squeezing-correlation parameters behave in a similar way : there are independent interchange fluxes of them proportional to their current values.

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